Tridiagonal matrices with nonnegative entries

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Abstract

In this paper we characterize the nonnegative irreducible tridiagonal matrices and their permutations, using certain entries in their primitive idempotents. Our main result is summarized as follows. Let d denote a nonnegative integer. Let A denote a matrix in $\mathrm{Mat}_{d+1}(\mathbb{R})$ and let $\{\theta_i\}_{i=0}^d$ denote the roots of the characteristic polynomial of A. We say A is *multiplicity-free* whenever these roots are mutually distinct and contained in $\mathbb{R}$. In this case E_i will denote the primitive idempotent of A associated with θ_i $(0 \leq i \leq d)$. We say A is *symmetrizable* whenever there exists an invertible diagonal matrix $\Delta \in \mathrm{Mat}_{d+1}(\mathbb{R})$ such that $\Delta A \Delta^{-1}$ is symmetric. Let $\Gamma(A)$ denote the directed graph with vertex set $\{0, 1, \ldots, d\}$, where $i \to j$ whenever $i \neq j$ and $A_{ij} \neq 0$.

Theorem. *Assume that each entry of A is nonnegative. Then the following are equivalent for $0 \leq s, t \leq d$.*

(i) *The graph $\Gamma(A)$ is a bidirected path with endpoints s, t:*

$$s \leftrightarrow * \leftrightarrow * \leftrightarrow \cdots \leftrightarrow * \leftrightarrow t.$$

(ii) *The matrix A is symmetrizable and multiplicity-free. Moreover the (s,t)-entry of E_i times*

$$(\theta_i - \theta_0) \cdots (\theta_i - \theta_{i-1})(\theta_i - \theta_{i+1}) \cdots (\theta_i - \theta_d)$$

is independent of i for $0 \leq i \leq d$, and this common value is nonzero.

Recently Kurihara and Nozaki obtained a theorem that characterizes the Q-polynomial property for symmetric association schemes. We view the above result as a linear algebraic generalization of their theorem.

1 Introduction

Recently Kurihara and Nozaki gave the following characterization of the Q-polynomial property for symmetric association schemes (see Section 4 for definitions).

Theorem 1.1 [3, Theorem 1.1] *Let $\mathcal{X}$ denote a d-class symmetric association scheme with adjacency matrices $\{A_i\}_{i=0}^d$. Let E and F denote primitive idempotents of $\mathcal{X}$ with E nontrivial. For $0 \leq i \leq d$ let θ^*_i denote the dual eigenvalue of E for A_i. Then the following are equivalent.*

(i) *$\mathcal{X}$ is Q-polynomial relative to E, and F is the last primitive idempotent in this Q-polynomial structure.*

(ii) *$\{\theta^*_i\}_{i=0}^d$ are mutually distinct, and for $0 \leq i \leq d$ the eigenvalue of A_i for F is*

$$\frac{(\theta^*_0 - \theta^*_1)(\theta^*_0 - \theta^*_2) \cdots (\theta^*_0 - \theta^*_d)}{(\theta^*_i - \theta^*_0) \cdots (\theta^*_i - \theta^*_{i-1})(\theta^*_i - \theta^*_{i+1}) \cdots (\theta^*_i - \theta^*_d)}. \tag{1}$$

As suggested by [3], there is a "dual" version of Theorem 1.1 in which the Q-polynomial structure is replaced by a P-polynomial structure. We now state this dual version.

Theorem 1.2 *Let $\mathcal{X}$ denote a d-class symmetric association scheme with primitive idempotents $\{E_i\}_{i=0}^d$. Let B and C denote adjacency matrices of $\mathcal{X}$ with B nontrivial. For $0 \leq i \leq d$ let θ_i denote the eigenvalue of B for E_i. Then the following are equivalent.*

(i) *$\mathcal{X}$ is P-polynomial relative to B, and C is the last adjacency matrix in this P-polynomial structure.*

(ii) $\{\theta_i\}_{i=0}^d$ are mutually distinct, and for $0 \leq i \leq d$ the dual eigenvalue of E_i for C is

$$\frac{(\theta_0 - \theta_1)(\theta_0 - \theta_2) \cdots (\theta_0 - \theta_d)}{(\theta_i - \theta_0) \cdots (\theta_i - \theta_{i-1})(\theta_i - \theta_{i+1}) \cdots (\theta_i - \theta_d)}. \tag{2}$$

In this paper we show that Theorems 1.1 and 1.2 follow from a linear algebraic result concerning matrices with nonnegative entries. We give two versions of the result, which are Theorems 1.3 and 1.4 below. Theorem 1.4 is the general version, and Theorem 1.3 is about an attractive special case. Before presenting these theorems we recall some concepts from linear algebra.

Throughout the paper $\mathbb{R}$ denotes the field of real numbers, d denotes a nonnegative integer, and $\mathrm{Mat}_{d+1}(\mathbb{R})$ denotes the $\mathbb{R}$-algebra consisting of the $(d+1) \times (d+1)$ matrices that have all entries in $\mathbb{R}$. We index the rows and columns by $0, 1, \ldots, d$. Let $V = \mathbb{R}^{d+1}$ denote the vector space over $\mathbb{R}$ consisting of the $(d+1)\times 1$ matrices that have all entries in $\mathbb{R}$. We index the rows by $0, 1, \ldots, d$. Observe that $\mathrm{Mat}_{d+1}(\mathbb{R})$ acts on V by left multiplication.

Let A denote a matrix in $\mathrm{Mat}_{d+1}(\mathbb{R})$. We say A is *nonnegative* whenever each entry of A is nonnegative. We say A is *symmetric* whenever $A^t = A$, where t denotes transpose. We say A is *symmetrizable* whenever there exists an invertible diagonal matrix $\Delta \in \mathrm{Mat}_{d+1}(\mathbb{R})$ such that $\Delta A \Delta^{-1}$ is symmetric. A subspace $W \subseteq V$ is called an *eigenspace* of A whenever $W \neq 0$ and there exists $\theta \in \mathbb{R}$ such that $W = \{v \in V \,|\, Av = \theta v\}$; in this case θ is the *eigenvalue* of A associated with W. We say A is *diagonalizable* whenever its eigenspaces span V. We say that A is *multiplicity-free* whenever A is diagonalizable and its eigenspaces all have dimension 1. Assume A is multiplicity-free and let $\{\theta_i\}_{i=0}^d$ denote an ordering of the eigenvalues of A. For $0 \leq i \leq d$ let V_i denote the eigenspace of A associated with θ_i. For $0 \leq i \leq d$ define $E_i \in \mathrm{Mat}_{d+1}(\mathbb{R})$ such that $(E_i - I)V_i = 0$ and $E_i V_j = 0$ for $j \neq i$ $(0 \leq j \leq d)$. Here I denotes the identity matrix in $\mathrm{Mat}_{d+1}(\mathbb{R})$. We call E_i the *primitive idempotent* of A associated with V_i (or θ_i). Observe that (i) $I = \sum_{i=0}^d E_i$; (ii) $E_i E_j = \delta_{i,j} E_i$ $(0 \leq i, j \leq d)$; (iii) $V_i = E_i V$ $(0 \leq i \leq d)$; (iv) $A = \sum_{i=0}^d \theta_i E_i$. Using these facts we find

$$E_i = \prod_{\substack{0 \leq j \leq d \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j} \qquad (0 \leq i \leq d). \tag{3}$$

Again let A denote a matrix in $\mathrm{Mat}_{d+1}(\mathbb{R})$. We say that A is *tridiagonal* whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. Assume for the moment that A is tridiagonal. Then A is said to be *irreducible* whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero. Let $\Gamma(A)$ denote the directed graph with vertex set $\{0, 1, \ldots, d\}$, where $i \to j$ whenever $i \neq j$ and $A_{ij} \neq 0$. Observe that the following are equivalent: (i) A is irreducible tridiagonal; (ii) $\Gamma(A)$ is the bidirected path $0 \leftrightarrow 1 \leftrightarrow 2 \leftrightarrow \cdots \leftrightarrow d$. More generally the following

are equivalent: (i) there exists a permutation matrix $\Lambda \in \mathrm{Mat}_{d+1}(\mathbb{R})$ such that $\Lambda A \Lambda^{-1}$ is irreducible tridiagonal; (ii) $\Gamma(A)$ is a bidirected path. We now state our first main result.

Theorem 1.3 *Let A denote a nonnegative matrix in $\mathrm{Mat}_{d+1}(\mathbb{R})$. Then the following are equivalent for $0 \leq s, t \leq d$.*

(i) *The graph $\Gamma(A)$ is a bidirected path with endpoints s, t:*

$$s \leftrightarrow * \leftrightarrow * \leftrightarrow \cdots \leftrightarrow * \leftrightarrow t.$$

(ii) *The matrix A is symmetrizable and multiplicity-free. Moreover the (s,t)-entry of E_i times*

$$(\theta_i - \theta_0) \cdots (\theta_i - \theta_{i-1})(\theta_i - \theta_{i+1}) \cdots (\theta_i - \theta_d)$$

is independent of i for $0 \leq i \leq d$, and this common value is nonzero.

The proof of Theorem 1.3 is given in Section 3. Before stating our second main result, we make a few comments. For $A \in \mathrm{Mat}_{d+1}(\mathbb{R})$ we say that A is *(upper) Hessenberg* whenever each entry below the subdiagonal is zero and each entry on the subdiagonal is nonzero. Observe that the following are equivalent: (i) A is Hessenberg; (ii) in the graph $\Gamma(A)$, for all vertices i, j we have $i \to j$ if $i - j = 1$ and $i \not\to j$ if $i - j > 1$. An ordering $\{x_i\}_{i=0}^d$ of the vertices of $\Gamma(A)$ is called *Hessenberg* whenever for $0 \leq i, j \leq d$, $x_i \to x_j$ if $i - j = 1$ and $x_i \not\to x_j$ if $i - j > 1$. We recall the directed distance function ∂ for $\Gamma(A)$. Given vertices s, t of $\Gamma(A)$ and an integer i $(0 \leq i \leq d)$, we have $\partial(s,t) = i$ whenever there exists a directed path in $\Gamma(A)$ from s to t that has length i, and there does not exist a directed path in $\Gamma(A)$ from s to t that has length less than i. For all vertices s, t in $\Gamma(A)$ the following are equivalent: (i) there exists a Hessenberg ordering $\{x_i\}_{i=0}^d$ of the vertices of $\Gamma(A)$ such that $x_0 = t$ and $x_d = s$; (ii) $\partial(s,t) = d$. We now state our second main result.

Theorem 1.4 *Let A denote a nonnegative matrix in $\mathrm{Mat}_{d+1}(\mathbb{R})$. Then the following are equivalent for $0 \leq s, t \leq d$.*

(i) *The matrix A is diagonalizable, and $\partial(s,t) = d$ in $\Gamma(A)$.*

(ii) *The matrix A is multiplicity-free. Moreover the (s,t)-entry of E_i times*

$$(\theta_i - \theta_0) \cdots (\theta_i - \theta_{i-1})(\theta_i - \theta_{i+1}) \cdots (\theta_i - \theta_d)$$

is independent of i for $0 \leq i \leq d$, and this common value is nonzero.

The proof of Theorem 1.4 is given in Section 2. In Sections 4–6 we apply Theorem 1.3 to symmetric association schemes. In Sections 4 and 5 we give some basic facts about these objects. In Section 6 we use these facts and Theorem 1.3 to prove Theorems 1.1 and 1.2.

2 Hessenberg matrices

In this section we prove Theorem 1.4.

Lemma 2.1 *Let A denote a Hessenberg matrix in* $\mathrm{Mat}_{d+1}(\mathbb{R})$*. Then for $0 \leq r \leq d$ the entries of A^r are described as follows. For $0 \leq i, j \leq d$ the (i,j)-entry is nonzero if $i-j=r$ and zero if $i-j>r$.*

Proof. Use matrix multiplication and the definition of Hessenberg. $\square$

Corollary 2.2 *Let A denote a Hessenberg matrix in* $\mathrm{Mat}_{d+1}(\mathbb{R})$*. Then the matrices $\{A^r\}_{r=0}^d$ are linearly independent.*

Proof. For $0 \leq r \leq d$ let $u_r \in \mathbb{R}^{d+1}$ denote the 0^{th} column of A^r. By Lemma 2.1 the i^{th} entry of u_r is nonzero for $i=r$ and zero for $r+1 \leq i \leq d$. Therefore $\{u_r\}_{r=0}^d$ are linearly independent. The result follows. $\square$

Lemma 2.3 *Let A denote a Hessenberg matrix in* $\mathrm{Mat}_{d+1}(\mathbb{R})$*. Then the minimal polynomial of A equals the characteristic polynomial of A.*

Proof. By construction the characteristic polynomial of A is monic with degree $d+1$. By elementary linear algebra the minimal polynomial of A is monic and divides the characteristic polynomial of A. By Corollary 2.2 the minimal polynomial of A has degree $d+1$. The result follows. $\square$

Lemma 2.4 *Let A denote a diagonalizable Hessenberg matrix in* $\mathrm{Mat}_{d+1}(\mathbb{R})$*. Then A is multiplicity-free.*

Proof. Let $\{\theta_i\}_{i=0}^d$ denote the roots of the characteristic polynomial of A. We have $\theta_i \in \mathbb{R}$ $(0 \leq i \leq d)$ since A is diagonalizable. Moreover $\{\theta_i\}_{i=0}^d$ are mutually distinct since the minimal polynomial of A equals the characteristic polynomial of A by Lemma 2.3 and since the roots of the minimal polynomial are mutually distinct. Thus A is multiplicity-free. $\square$

For $A \in \mathrm{Mat}_{d+1}(\mathbb{R})$ let $\Gamma_\ell(A)$ denote the directed graph with vertex set $\{0, 1, \ldots, d\}$, where $i \to j$ whenever $A_{ij} \neq 0$.

Lemma 2.5 *Let A denote a nonnegative matrix in* $\mathrm{Mat}_{d+1}(\mathbb{R})$*. Then the following are equivalent for $0 \leq r, s, t \leq d$.*

(i) *The (s,t)-entry of A^r is nonzero.*

(ii) *In the graph $\Gamma_\ell(A)$ there exists a directed path of length r from s to t.*

Proof. Consider the (s,t)-entry of A^r using matrix multiplication. $\square$

Lemma 2.6 *The following are equivalent for all $A \in \mathrm{Mat}_{d+1}(\mathbb{R})$ and $0 \le r, s, t \le d$.*

(i) $\partial(s,t) = r$ *in* $\Gamma(A)$.

(ii) $\partial(s,t) = r$ *in* $\Gamma_\ell(A)$.

Proof. Routine verification. $\square$

Lemma 2.7 *Let A denote a nonnegative matrix in $\mathrm{Mat}_{d+1}(\mathbb{R})$. Then the following are equivalent for $0 \le s, t \le d$.*

(i) *The (s,t)-entry of A^r is nonzero if $r = d$ and zero if $r < d$ $(0 \le r \le d)$.*

(ii) $\partial(s,t) = d$ *in* $\Gamma(A)$.

Proof. Follows from Lemmas 2.5 and 2.6 $\square$

Let λ denote an indeterminate and let $\mathbb{R}[\lambda]$ denote the $\mathbb{R}$-algebra consisting of the polynomials in λ that have all coefficients in $\mathbb{R}$.

Lemma 2.8 *For $0 \le i \le d$ let $f_i \in \mathbb{R}[\lambda]$ be monic with degree d, and assume $\{f_i\}_{i=0}^d$ are linearly independent. Then the following are equivalent for all $A \in \mathrm{Mat}_{d+1}(\mathbb{R})$ and $0 \le s, t \le d$.*

(i) *The (s,t)-entry of A^r is zero for $0 \le r \le d-1$.*

(ii) *The (s,t)-entry of $f_i(A)$ is equal to the (s,t)-entry of A^d for $0 \le i \le d$.*

(iii) *The (s,t)-entry of $f_i(A)$ is independent of i for $0 \le i \le d$.*

Proof. (i)$\Rightarrow$(ii): Since f_i is monic with degree d.

(ii)$\Rightarrow$(iii): Clear.

(iii)$\Rightarrow$(i): For $1 \le i \le d$ define $g_i = f_i - f_0$ and observe that g_i has degree at most $d-1$. Note that $\{g_i\}_{i=1}^d$ are linearly independent. So $\{g_i\}_{i=1}^d$ form a basis for the subspace of $\mathbb{R}[\lambda]$ consisting of the polynomials with degree at most $d-1$. So for $0 \le r \le d-1$, λ^r is a linear combination of $\{g_i\}_{i=1}^d$. By construction the (s,t)-entry of $g_i(A)$ is zero for $1 \le i \le d$. By these comments the (s,t)-entry of A^r is zero for $0 \le r \le d-1$. $\square$

Referring to Lemma 2.8 we now make a specific choice for the polynomials $\{f_i\}_{i=0}^d$.

Lemma 2.9 *Assume $A \in \mathrm{Mat}_{d+1}(\mathbb{R})$ is multiplicity-free with eigenvalues $\{\theta_i\}_{i=0}^d$. For $0 \le i \le d$ define a polynomial $f_i \in \mathbb{R}[\lambda]$ by*

$$f_i = (\lambda - \theta_0) \cdots (\lambda - \theta_{i-1})(\lambda - \theta_{i+1}) \cdots (\lambda - \theta_d). \tag{4}$$

Then

(i) $f_i(A) = f_i(\theta_i) E_i$ *for* $0 \le i \le d$.

(ii) f_i *is monic with degree d for $0 \le i \le d$.*

(iii) $\{f_i\}_{i=0}^d$ *are linearly independent.*

Proof. (i): Compare (3) and (4).

(ii): Clear.

(iii): For $0 \leq i, j \leq d$ the scalar $f_i(\theta_j)$ is zero if $i \neq j$ and nonzero if $i = j$. $\square$

Proof of Theorem 1.4.

(i)⇒(ii): By the comments above Theorem 1.4 there exists a Hessenberg ordering $\{x_i\}_{i=0}^d$ of the vertices of $\Gamma(A)$ such that $x_0 = t$ and $x_d = s$. Let $\Lambda \in \mathrm{Mat}_{d+1}(\mathbb{R})$ denote the permutation matrix that corresponds to the permutation $i \mapsto x_i$ $(0 \leq i \leq d)$. Then $\Lambda A \Lambda^{-1}$ is Hessenberg. We assume A is diagonalizable so $\Lambda A \Lambda^{-1}$ is diagonalizable. Now $\Lambda A \Lambda^{-1}$ is multiplicity-free by Lemma 2.4 so A is multiplicity-free. Define the polynomials $\{f_i\}_{i=0}^d$ as in Lemma 2.9. By Lemma 2.7, for $0 \leq r \leq d$ the (s,t)-entry of A^r is nonzero if $r = d$ and zero if $r < d$. By this and Lemma 2.8, the (s,t)-entry of $f_i(A)$ is independent of i for $0 \leq i \leq d$, and this common value is nonzero. By this and Lemma 2.9(i), the (s,t)-entry of E_i times $f_i(\theta_i)$ is independent of i for $0 \leq i \leq d$, and this common value is nonzero.

(ii)⇒(i): The matrix A is diagonalizable since it is multiplicity-free. Define $\{f_i\}_{i=0}^d$ as in Lemma 2.9. By assumption, the (s,t)-entry of E_i times $f_i(\theta_i)$ is independent of i for $0 \leq i \leq d$, and this common value is nonzero. By this and Lemma 2.9(i), the (s,t)-entry of $f_i(A)$ is independent of i for $0 \leq i \leq d$, and this common value is nonzero. By this and Lemma 2.8, for $0 \leq r \leq d$ the (s,t)-entry of A^r is nonzero if $r = d$ and zero if $r < d$. By this and Lemma 2.7 we find $\partial(s,t) = d$. $\square$

3 Tridiagonal matrices

In this section we prove Theorem 1.3.

Lemma 3.1 *Let A denote a symmetrizable matrix in $\mathrm{Mat}_{d+1}(\mathbb{R})$. Then $A_{ij} = 0$ if and only if $A_{ji} = 0$ $(0 \leq i, j \leq d)$.*

Proof. Since A is symmetrizable, there exists an invertible diagonal matrix $\Delta \in \mathrm{Mat}_{d+1}(\mathbb{R})$ such that $\Delta A \Delta^{-1}$ is symmetric. Comparing the (i,j)-entry and the (j,i)-entry of $\Delta A \Delta^{-1}$ we find $\Delta_{ii} A_{ij} \Delta_{jj}^{-1} = \Delta_{jj} A_{ji} \Delta_{ii}^{-1}$. The result follows. $\square$

Lemma 3.2 *Let A denote a symmetrizable matrix in $\mathrm{Mat}_{d+1}(\mathbb{R})$. Then A is diagonalizable.*

Proof. Since A is symmetrizable, there exists an invertible diagonal matrix $\Delta \in \mathrm{Mat}_{d+1}(\mathbb{R})$ such that $\Delta A \Delta^{-1}$ symmetric. By [4, Corollary 3.3.1] every symmetric matrix in $\mathrm{Mat}_{d+1}(\mathbb{R})$ is diagonalizable. Therefore $\Delta A \Delta^{-1}$ is diagonalizable, so A is diagonalizable. $\square$

Lemma 3.3 *Let $A \in \mathrm{Mat}_{d+1}(\mathbb{R})$ denote a symmetrizable matrix. Then $\Lambda A \Lambda^{-1}$ is symmetrizable for every permutation matrix $\Lambda \in \mathrm{Mat}_{d+1}(\mathbb{R})$.*

Proof. Since A is symmetrizable, there exists an invertible diagonal matrix $\Delta \in \mathrm{Mat}_{d+1}(\mathbb{R})$ such that $\Delta A \Delta^{-1}$ is symmetric. Set $\Delta' = \Lambda \Delta \Lambda^{-1}$, and observe that Δ' is invertible diagonal. Using $\Lambda^{-1} = \Lambda^t$ we find $\Delta' \Lambda A \Lambda^{-1} (\Delta')^{-1}$ is symmetric. Now $\Lambda A \Lambda^{-1}$ is symmetrizable. $\square$

Lemma 3.4 *Let A denote a nonnegative irreducible tridiagonal matrix in* $\mathrm{Mat}_{d+1}(\mathbb{R})$*. Then A is symmetrizable and multiplicity-free.*

Proof. We first show that A is symmetrizable. Since A is irreducible and nonnegative we have $A_{i,i-1} > 0$ and $A_{i-1,i} > 0$ for $1 \leq i \leq d$. For $0 \leq i \leq d$ define

$$\kappa_i = \frac{A_{01}A_{12}\cdots A_{i-1,i}}{A_{10}A_{21}\cdots A_{i,i-1}}$$

and note that $\kappa_i > 0$. Define a diagonal matrix $K \in \mathrm{Mat}_{d+1}(\mathbb{R})$ with (i,i)-entry κ_i for $0 \leq i \leq d$. Using matrix multiplication one finds $KA = A^t K$. Define a diagonal matrix $\Delta \in \mathrm{Mat}_{d+1}(\mathbb{R})$ with (i,i)-entry $\sqrt{\kappa_i}$ for $0 \leq i \leq d$, so that $\Delta^2 = K$. By this and $KA = A^t K$ one finds that $\Delta A \Delta^{-1}$ is symmetric. Therefore A is symmetrizable. Now A is diagonalizable by Lemma 3.2 and multiplicity-free by Lemma 2.4. $\square$

Lemma 3.5 *Let A denote a nonnegative matrix in* $\mathrm{Mat}_{d+1}(\mathbb{R})$*. Then the following are equivalent for* $0 \leq s,t \leq d$*.*

(i) *The graph $\Gamma(A)$ is a bidirected path with endpoints s, t.*

(ii) *The matrix A is symmetrizable, and $\partial(s,t) = d$ in $\Gamma(A)$.*

Proof. (i)$\Rightarrow$(ii): We first show that A is symmetrizable. By the observation above Theorem 1.3, there exists a permutation matrix $\Lambda \in \mathrm{Mat}_{d+1}(\mathbb{R})$ such that $\Lambda A \Lambda^{-1}$ is irreducible tridiagonal. We assume A is nonnegative so $\Lambda A \Lambda^{-1}$ is nonnegative. So $\Lambda A \Lambda^{-1}$ is symmetrizable in vew of Lemma 3.4. Now A is symmetrizable by Lemma 3.3. By construction $\partial(s,t) = d$ in $\Gamma(A)$.

(ii)$\Rightarrow$(i): Routine using Lemma 3.1. $\square$

Proof of Theorem 1.3. (i)$\Rightarrow$(ii): A is symmetrizable by Lemma 3.5, and A is diagonalizable by Lemma 3.2. By Lemma 3.5, $\partial(s,t) = d$ in $\Gamma(A)$. The result follows in view of Theorem 1.4.

(ii)$\Rightarrow$(i): By Theorem 1.4 $\partial(s,t) = d$ in $\Gamma(A)$. By this and Lemma 3.5 the graph $\Gamma(A)$ is a bidirected path with endpoints s, t. $\square$

4 Symmetric association schemes

In this section we review some definitions and basic concepts concerning symmetric association schemes. For more information we refer the reader to [1, 2, 5].

A *d-class symmetric association scheme* is a pair $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$, where X is a finite nonempty set and $\{R_i\}_{i=0}^d$ are nonempty subsets of $X \times X$ that satisfy

(i) $R_0 = \{(x,x) \mid x \in X\}$;

(ii) $X \times X = R_0 \cup R_1 \cup \cdots \cup R_d$ (disjoint union);

(iii) $R_i^t = R_i$ for $0 \leq i \leq d$, where $R_i^t = \{(y,x) \mid (x,y) \in R_i\}$;

(iv) there exist integers p^h_{ij} $(0 \le h, i, j \le d)$ such that, for every $(x, y) \in R_h$,

$$p^h_{ij} = |\{z \in X \mid (x, z) \in R_i,\ (z, y) \in R_j\}|.$$

The parameters p^h_{ij} are called the *intersection numbers* of $\mathcal{X}$.

From now on let $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ denote a d-class symmetric association scheme. Observe by (iii) that $p^h_{ij} = p^h_{ji}$ for $0 \le h, i, j \le d$. For $0 \le i \le d$ define $k_i = p^0_{ii}$, and observe

$$k_i = |\{y \in X \mid (x, y) \in R_i\}| \qquad (x \in X).$$

Note that $k_i > 0$. By [1, Proposition II.2.2],

$$k_h p^h_{ij} = k_j p^j_{ih} \qquad (0 \le h, i, j \le d). \tag{5}$$

We recall the Bose-Mesner algebra of $\mathcal{X}$. Let $\mathrm{Mat}_X(\mathbb{R})$ denote the $\mathbb{R}$-algebra consisting of the matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{R}$. For $0 \le i \le d$ let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{R})$ with (x, y)-entry

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } (x, y) \in R_i \\ 0 & \text{if } (x, y) \notin R_i \end{cases} \qquad (x, y \in X).$$

We call $\{A_i\}_{i=0}^d$ the *adjacency matrices* of $\mathcal{X}$. Note that $A_0 = I$, where I denotes the identity matrix in $\mathrm{Mat}_X(\mathbb{R})$. We call A_0 the *trivial* adjacency matrix. Observe $A_i^t = A_i$ for $0 \le i \le d$. The matrices $\{A_i\}_{i=0}^d$ are linearly independent since they have nonzero entries which are in disjoint positions. Observe

$$A_i A_j = \sum_{h=0}^d p^h_{ij} A_h \qquad (0 \le i, j \le d). \tag{6}$$

By $p^h_{ij} = p^h_{ji}$ we find $A_i A_j = A_j A_i$ for $0 \le i, j \le d$. Using these facts we find $\{A_i\}_{i=0}^d$ is a basis for a commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{R})$. We call M the *Bose-Mesner algebra* of $\mathcal{X}$.

By [1, Section II.2.3] M has a second basis $\{E_i\}_{i=0}^d$ such that (i) $E_0 = |X|^{-1}J$; (ii) $I = \sum_{i=0}^d E_i$; (iii) $E_i^t = E_i$ $(0 \le i \le d)$; (iv) $E_i E_j = \delta_{i,j} E_i$ $(0 \le i, j \le d)$. We call $\{E_i\}_{i=0}^d$ the *primitive idempotents* of $\mathcal{X}$. We call E_0 the *trivial* primitive idempotent. For $0 \le i \le d$ let m_i denote the rank of E_i. Note that $m_i > 0$.

We recall the matrices P and Q. We mentioned above that $\{A_i\}_{i=0}^d$ and $\{E_i\}_{i=0}^d$ are bases for M. Define $P \in \mathrm{Mat}_{d+1}(\mathbb{R})$ such that

$$A_j = \sum_{i=0}^d P_{ij} E_i \qquad (0 \le j \le d). \tag{7}$$

Define $Q \in \mathrm{Mat}_{d+1}(\mathbb{R})$ such that

$$E_j = |X|^{-1} \sum_{i=0}^d Q_{ij} A_i \qquad (0 \le j \le d). \tag{8}$$

Observe that $PQ = QP = |X|I$. Setting $j = 0$ and $A_0 = I$ in (7) we find $P_{i0} = 1$ for $0 \leq i \leq d$. Setting $j = 0$ and $E_0 = |X|^{-1}J$ in (8) we find $Q_{i0} = 1$ for $0 \leq i \leq d$.

We recall the P-polynomial property. Let $\{A_i\}_{i=1}^d$ denote an ordering of the nontrivial adjacency matrices of $\mathcal{X}$. This ordering is said to be *P-polynomial* whenever for $0 \leq i, j \leq d$ the intersection number p^1_{ij} is zero if $|i - j| > 1$ and nonzero if $|i - j| = 1$. Let A denote a nontrivial adjacency matrix of $\mathcal{X}$. We say $\mathcal{X}$ is *P-polynomial relative to A* whenever there exists a P-polynomial ordering $\{A_i\}_{i=1}^d$ of the nontrivial adjacency matrices such that $A_1 = A$. In this case we call A_d the *last adjacency matrix* in this P-polynomial structure.

We recall the Krein parameters. Let $\circ$ denote the entrywise product in $\mathrm{Mat}_X(\mathbb{R})$. Observe $A_i \circ A_j = \delta_{i,j} A_i$ for $0 \leq i, j \leq d$, so M is closed under $\circ$. Thus there exist $q^h_{ij} \in \mathbb{R}$ $(0 \leq h, i, j \leq d)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{d} q^h_{ij} E_h \qquad (0 \leq i, j \leq d). \tag{9}$$

The parameters q^h_{ij} are called the *Krein parameters* of $\mathcal{X}$. By [1, Theorem II.3.8] the Krein parameters are nonnegative. By (9) we have $q^h_{ij} = q^h_{ji}$ for $0 \leq h, i, j \leq d$. Setting $j = 0$ and $E_0 = |X|^{-1}J$ in (9) we find $q^h_{i0} = \delta_{h,i}$ for $0 \leq h, i \leq d$. By [1, Proposition II.3.7],

$$m_h q^h_{ij} = m_j q^j_{ih} \qquad (0 \leq h, i, j \leq d). \tag{10}$$

We recall the Q-polynomial property. Let $\{E_i\}_{i=1}^d$ denote an ordering of the nontrivial primitive idempotents of $\mathcal{X}$. This ordering is said to be *Q-polynomial* whenever for $0 \leq i, j \leq d$ the Krein parameter q^1_{ij} is zero if $|i - j| > 1$ and nonzero if $|i - j| = 1$. Let E denote a nontrivial primitive idempotent of $\mathcal{X}$. We say $\mathcal{X}$ is *Q-polynomial relative to E* whenever there exists a Q-polynomial ordering $\{E_i\}_{i=1}^d$ of the nontrivial primitive idempotents such that $E_1 = E$. In this case we call E_d the *last primitive idempotent* in this Q-polynomial structure.

We recall the dual Bose-Mesner algebra. For the rest of the paper fix $x \in X$. For $0 \leq i \leq d$ let E^*_i denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{R})$ with (y, y)-entry

$$(E^*_i)_{yy} = \begin{cases} 1 & \text{if } (x, y) \in R_i \\ 0 & \text{if } (x, y) \notin R_i \end{cases} \qquad (y \in X).$$

For $y \in X$ the (y, y)-entry of E^*_i coincides with the (x, y)-entry of A_i. Observe $E^*_i E^*_j = \delta_{i,j} E^*_i$ $(0 \leq i, j \leq d)$ and $I = \sum_{i=0}^d E^*_i$. We call $\{E^*_i\}_{i=0}^d$ the *dual primitive idempotents* of $\mathcal{X}$. By the above comments $\{E^*_i\}_{i=0}^d$ is a basis for a commutative subalgebra M^* of $\mathrm{Mat}_X(\mathbb{R})$. We call M^* the *dual Bose-Mesner algebra* of $\mathcal{X}$. For $0 \leq i \leq d$ let A^*_i denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{R})$ with (y, y)-entry $|X|(E_i)_{x,y}$ for $y \in X$. We call $\{A^*_i\}_{i=0}^d$ the *dual adjacency matrices* of $\mathcal{X}$. Using (8),

$$A^*_j = \sum_{i=0}^{d} Q_{ij} E^*_i \qquad (0 \leq j \leq d). \tag{11}$$

Using (7),

$$E_j^* = |X|^{-1} \sum_{i=0}^{d} P_{ij} A_i^* \qquad (0 \le j \le d). \tag{12}$$

Using (9),

$$A_i^* A_j^* = \sum_{h=0}^{d} q_{ij}^h A_h^* \qquad (0 \le i, j \le d). \tag{13}$$

For $0 \le i, j \le d$ the scalar P_{ij} (resp. Q_{ij}) is known as the *eigenvalue* of A_j for E_i (resp. *dual eigenvalue* of E_j for A_i).

5 The subconstituent algebra and its primary module

We continue to discuss the symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ from Section 4. Let $\mathbb{R}^X$ denote the vector space over $\mathbb{R}$ consisting of column vectors with entries in $\mathbb{R}$ and coordinates indexed by X. Observe that $\mathrm{Mat}_X(\mathbb{R})$ acts on $\mathbb{R}^X$ by left multiplication. For all $y \in X$ let $\hat{y}$ denote the vector in $\mathbb{R}^X$ that has y-coordinate 1 and all other coordinates 0. Note that $\{\hat{y} \,|\, y \in X\}$ is a basis for $\mathbb{R}^X$. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{R})$ generated by M and M^*. We call T the *subconstituent algebra* of $\mathcal{X}$ with respect to x [5, Definition 3.3]. We now describe a certain irreducible T-module known as the primary module. Let $\mathbf{1} = \sum_{y \in X} \hat{y}$ denote the "all 1's" vector in $\mathbb{R}^X$. By construction, for $0 \le i \le d$ we have $E_i^* \mathbf{1} = A_i \hat{x}$ and $A_i^* \mathbf{1} = |X| E_i \hat{x}$. Therefore $M\hat{x} = M^* \mathbf{1}$. Denote this common space by W and observe that W is a T-module. This T-module is said to be *primary.* The T-module W is irreducible by [5, Lemma 3.6]. We now describe two bases for W. For $0 \le i \le d$ define

$$\mathbf{1}_i = E_i^* \mathbf{1} = A_i \hat{x}, \qquad \mathbf{1}_i^* = A_i^* \mathbf{1} = |X| E_i \hat{x}. \tag{14}$$

Then each of $\{\mathbf{1}_i\}_{i=0}^d$ and $\{\mathbf{1}_i^*\}_{i=0}^d$ is a basis for W. We now describe the transition matrices between these bases. Using (7) and (14) we find

$$\mathbf{1}_j = |X|^{-1} \sum_{i=0}^{d} P_{ij} \mathbf{1}_i^* \qquad (0 \le j \le d). \tag{15}$$

By (15) and $PQ = |X| I$,

$$\mathbf{1}_j^* = \sum_{i=0}^{d} Q_{ij} \mathbf{1}_i \qquad (0 \le j \le d). \tag{16}$$

Definition 5.1 For all $B \in T$ let $\rho(B) \in \mathrm{Mat}_{d+1}(\mathbb{R})$ denote the matrix that represents B with respect to the basis $\{\mathbf{1}_i\}_{i=0}^d$. Thus

$$B \mathbf{1}_j = \sum_{i=0}^{d} \rho(B)_{ij} \mathbf{1}_i \qquad (0 \le j \le d). \tag{17}$$

This defines an $\mathbb{R}$-algebra homomorphism $\rho : T \to \mathrm{Mat}_{d+1}(\mathbb{R})$.

Definition 5.2 For all $B \in T$ let $\rho^*(B) \in \mathrm{Mat}_{d+1}(\mathbb{R})$ denote the matrix that represents B with respect to the basis $\{\mathbf{1}_i^*\}_{i=0}^d$. Thus

$$B\mathbf{1}_j^* = \sum_{i=0}^{d} \rho^*(B)_{ij}\mathbf{1}_i^* \qquad (0 \leq j \leq d). \tag{18}$$

This defines an $\mathbb{R}$-algebra homomorphism $\rho^* : T \to \mathrm{Mat}_{d+1}(\mathbb{R})$.

Lemma 5.3 *The following hold for all $B \in T$.*

(i) $P\rho(B)P^{-1} = \rho^*(B)$.

(ii) $Q\rho^*(B)Q^{-1} = \rho(B)$.

Proof. (i): By (15) and elementary linear algebra.
(ii): Follows from (i) and $PQ = |X|I$. □

Lemma 5.4 *The following hold for $0 \leq h, i, j \leq d$.*

(i) $\rho(A_i)$ has (h,j)-entry p_{ij}^h.

(ii) $\rho^*(A_i^*)$ has (h,j)-entry q_{ij}^h.

Proof. (i): Using (6) and (14) we argue $A_i\mathbf{1}_j = A_iA_j\hat{x} = \sum_{h=0}^d p_{ij}^h A_h\hat{x} = \sum_{h=0}^d p_{ij}^h \mathbf{1}_h$.
(ii): Similar to the proof of (i). □

Lemma 5.5 *The following hold for $0 \leq i \leq d$.*

(i) *$\rho(A_i^*)$ is diagonal with (j,j)-entry Q_{ji} for $0 \leq j \leq d$.*

(ii) *$\rho^*(A_i)$ is diagonal with (j,j)-entry P_{ji} for $0 \leq j \leq d$.*

Proof. (i): Using (14) we argue $A_i^*\mathbf{1}_j = A_i^*E_j^*\mathbf{1} = Q_{ji}E_j^*\mathbf{1} = Q_{ji}\mathbf{1}_j$.
(ii): Similar to the proof of (i). □

Lemma 5.6 *The following hold for $0 \leq i \leq d$.*

(i) *$\rho(E_i^*)$ has (i,i)-entry 1 and all other entries 0.*

(ii) *$\rho^*(E_i)$ has (i,i)-entry 1 and all other entries 0.*

Proof. (i): Using (14) we argue $E_i^*\mathbf{1}_j = E_i^*E_j^*\mathbf{1} = \delta_{i,j}E_i^*\mathbf{1} = \delta_{i,j}\mathbf{1}_i$.
(ii): Similar to the proof of (i). □

Lemma 5.7 *The following hold for $0 \leq h, i, j \leq d$.*

(i) *$\rho(E_i)$ has (h, j)-entry $|X|^{-1}Q_{hi}P_{ij}$.*

(ii) *$\rho^*(E_i^*)$ has (h, j)-entry $|X|^{-1}P_{hi}Q_{ij}$.*

Proof. (i): By Lemma 5.3(ii) $\rho(E_i) = Q\rho^*(E_i)Q^{-1}$. By $PQ = |X|I$ we have $Q^{-1} = |X|^{-1}P$. By Lemma 5.6(ii) $\rho^*(E_i)$ has (i, i)-entry 1 and all other entries 0. The result follows from these comments.

(ii): Similar to the proof of (i). □

Lemma 5.8 *For $0 \leq i \leq d$ each of $\rho(A_i)$ and $\rho^*(A_i^*)$ is nonnegative and symmetrizable.*

Proof. Concerning $\rho(A_i)$, observe it is nonnegative by Lemma 5.4(i). Define a diagonal matrix $\Delta \in \mathrm{Mat}_{d+1}(\mathbb{R})$ with (i, i)-entry $\sqrt{k_i}$ for $0 \leq i \leq d$. Using (5) we routinely find that $\Delta\rho(A_i)\Delta^{-1}$ is symmetric. Therefore $\rho(A_i)$ is symmetrizable. The proof for $\rho^*(A_i^*)$ is similar using (10) and Lemma 5.4(ii). □

We will need the following well-known facts.

Lemma 5.9 [1, Proposition III.1.1] *The following are equivalent.*

(i) *The ordering $\{A_i\}_{i=1}^d$ is P-polynomial.*

(ii) *The matrix $\rho(A_1)$ is irreducible tridiagonal.*

(iii) *The graph $\Gamma(\rho(A_1))$ is the bidirected path $0 \leftrightarrow 1 \leftrightarrow 2 \leftrightarrow \cdots \leftrightarrow d$.*

Proof. In the definition of P-polynomial, p^1_{ij} is nonzero if and only if p^i_{1j} is nonzero by (5) and since $k_h \neq 0$. Now we obtain the result using Lemma 5.4(ii). □

Lemma 5.10 [1, Section III.1] *The following are equivalent.*

(i) *The ordering $\{E_i\}_{i=1}^d$ is Q-polynomial.*

(ii) *The matrix $\rho^*(A_1^*)$ is irreducible tridiagonal.*

(iii) *The graph $\Gamma(\rho^*(A_1^*))$ is the bidirected path $0 \leftrightarrow 1 \leftrightarrow 2 \leftrightarrow \cdots \leftrightarrow d$.*

Proof. In the definition of Q-polynomial, q^1_{ij} is nonzero if and only if q^i_{1j} is nonzero by (10) and since $m_h \neq 0$. Now we obtain the result using Lemma 5.4(ii). □

Lemma 5.11 *For $1 \leq i \leq d$ consider the graph $\Gamma(\rho(A_i))$ and $\Gamma(\rho^*(A_i^*))$. In either case,*

(i) *$h \to j$ if and only if $j \to h$ $(0 \leq h, j \leq d)$.*

(ii) *$0 \to i$.*

(iii) *$0 \not\to h$ if $h \neq i$ $(0 \leq h \leq d)$.*

Proof. (i): By Lemmas 3.1 and 5.8.

(ii), (iii): By Lemma 5.4 and since $p^0_{ih} = \delta_{h,i}$, $q^0_{ih} = \delta_{h,i}$. □

Lemma 5.12 *For $1 \leq i \leq d$ the following are equivalent.*

(i) $\mathcal{X}$ *is P-polynomial relative to A_i.*

(ii) *The graph $\Gamma(\rho(A_i))$ is a bidirected path.*

Suppose (i) *and* (ii) *hold. Then the above graph is $0 \leftrightarrow i \leftrightarrow * \leftrightarrow * \cdots * \leftrightarrow s$, where A_s is the last adjacency matrix in the P-polynomial structure.*

Proof. Use Lemmas 5.9 and 5.11. □

Lemma 5.13 *For $1 \leq i \leq d$ the following are equivalent.*

(i) $\mathcal{X}$ *is Q-polynomial relative to E_i.*

(ii) *The graph $\Gamma(\rho^*(A_i^*))$ is a bidirected path.*

Suppose (i) *and* (ii) *hold. Then the above graph is $0 \leftrightarrow i \leftrightarrow * \leftrightarrow * \cdots * \leftrightarrow s$, where E_s is the last primitive idempotent in the Q-polynomial structure.*

Proof. Use Lemmas 5.10 and 5.11. □

6 Proof of Theorems 1.1 and 1.2

For convenience we first prove Theorem 1.2.

Proof of Theorem 1.2. Fix an ordering $\{A_i\}_{i=1}^d$ of the nontrivial adjacency matrices such that $A_1 = B$, and let $C = A_s$. For $0 \leq i \leq d$ the scalar $\theta_i = P_{i1}$ is the eigenvalue of B for E_i, and Q_{si} is the dual eigenvalue of E_i for C. For $0 \leq i \leq d$ define a polynomial $f_i \in \mathbb{R}[\lambda]$ by (4). Let the map ρ be as in Definition 5.1. By Lemma 5.8 the matrix $\rho(B)$ is nonnegative, so we can apply Theorem 1.3 with $A = \rho(B)$. We will do this after a few comments. Combining Lemma 5.3(i) and Lemma 5.5(ii) we find $P\rho(B)P^{-1} = \mathrm{diag}(\theta_0, \theta_1, \ldots, \theta_d)$. Therefore $\rho(B)$ is multiplicity-free if only if $\{\theta_i\}_{i=0}^d$ are mutually distinct, and in this case $\rho(E_i)$ is the primitive idempotent of $\rho(B)$ for θ_i $(0 \leq i \leq d)$. For $0 \leq i \leq d$ the $(s, 0)$-entry of $\rho(E_i)$ is given in Lemma 5.7(i). This entry is $|X|^{-1}Q_{si}$ since $P_{i0} = 1$.

(i)⇒(ii): By Lemma 5.12 the graph $\Gamma(\rho(B))$ is a bidirected path with endpoints s, 0. Therefore $A = \rho(B)$ satisfies Theorem 1.3(i) with $t = 0$. Applying Theorem 1.3 we draw two conclusions. First, $\rho(B)$ is multiplicity-free, so $\{\theta_i\}_{i=0}^d$ are mutually distinct. Second, the $(s, 0)$-entry of of $\rho(E_i)$ times $f_i(\theta_i)$ is independent of i for $0 \leq i \leq d$. By this and our above comments, $f_i(\theta_i)Q_{si}$ is independent of i for $0 \leq i \leq d$. Therefore $f_i(\theta_i)Q_{si} = f_0(\theta_0)Q_{s0}$ for $0 \leq i \leq d$. By this and since $Q_{s0} = 1$, we find $Q_{si} = f_0(\theta_0)f_i(\theta_i)^{-1}$ for $0 \leq i \leq d$. In other words, for $0 \leq i \leq d$ the dual eigenvalue of E_i for C is equal to (2).

(ii)⇒(i): Observe that $\rho(B)$ is symmetrizable by Lemma 5.8, and multiplicity-free since $\{\theta_i\}_{i=0}^d$ are mutually distinct. For $0 \leq i \leq d$ the dual eigenvalue of E_i for C is Q_{si}, and this is equal to $f_0(\theta_0)f_i(\theta_i)^{-1}$ by (2). By this and our above comments, for $0 \leq i \leq d$ the $(s, 0)$-entry of $\rho(E_i)$ is equal to $|X|^{-1}f_0(\theta_0)f_i(\theta_i)^{-1}$. So the $(s, 0)$-entry of $\rho(E_i)$ times $f_i(\theta_i)$ is independent of i for $0 \leq i \leq d$, and this common value is nonzero. Therefore

$\rho(B)$ satisfies Theorem 1.3(ii) with $t = 0$. Now by Theorem 1.3, the graph $\Gamma(\rho(B))$ is a bidirected path with endpoints s, 0. Now by Lemma 5.12 $\mathcal{X}$ is P-polynomial relative to B, and $C = A_s$ is the last adjacency matrix in this P-polynomial structure. $\square$

The proof of Theorem 1.1 is similar to the proof of Theorem 1.2. We give a precise proof for completeness.

Proof of Theorem 1.1. Fix an ordering $\{E_i\}_{i=1}^d$ of the nontrivial primitive idempotents such that $E_1 = E$, and let $F = E_s$. For $0 \le i \le d$ the scalar $\theta_i^* = Q_{i1}$ is the dual eigenvalue of E for A_i, and P_{si} is the eigenvalue of A_i for F. For $0 \le i \le d$ define a polynomial $f_i^* \in \mathbb{R}[\lambda]$ by

$$f_i^* = (\lambda - \theta_0^*) \cdots (\lambda - \theta_{i-1}^*)(\lambda - \theta_{i+1}^*) \cdots (\lambda - \theta_d^*).$$

For $0 \le i \le d$ let E_i^* (resp. A_i^*) denote the dual primitive idempotent (resp. dual adjacency matrix) corresponding to A_i (resp. E_i). Let the map ρ^* be as in Definition 5.2. By Lemma 5.8 the matrix $\rho^*(A_1^*)$ is nonnegative, so we can apply Theorem 1.3 with $A = \rho^*(A_1^*)$. We will do this after a few comments. Combining Lemma 5.3(ii) and Lemma 5.5(i) we find $Q\rho^*(A_1^*)Q^{-1} = \operatorname{diag}(\theta_0^*, \theta_1^*, \ldots, \theta_d^*)$. Therefore $\rho^*(A_1^*)$ is multiplicity-free if only if $\{\theta_i^*\}_{i=0}^d$ are mutually distinct, and in this case $\rho^*(E_i^*)$ is the primitive idempotent of $\rho^*(A_1^*)$ for θ_i^* $(0 \le i \le d)$. For $0 \le i \le d$ the $(s, 0)$-entry of $\rho^*(E_i^*)$ is given in Lemma 5.7(ii). This entry is $|X|^{-1}P_{si}$ since $Q_{i0} = 1$.

(i)⇒(ii): By Lemma 5.13 the graph $\Gamma(\rho^*(A_1^*))$ is a bidirected path with endpoints s, 0. Therefore $A = \rho^*(A_1^*)$ satisfies Theorem 1.3(i) with $t = 0$. Applying Theorem 1.3 we draw two conclusions. First, $\rho^*(A_1^*)$ is multiplicity-free, so $\{\theta_i^*\}_{i=0}^d$ are mutually distinct. Second, the $(s, 0)$-entry of of $\rho^*(E_i^*)$ times $f_i^*(\theta_i^*)$ is independent of i for $0 \le i \le d$. By this and our above comments, $f_i^*(\theta_i^*)P_{si}$ is independent of i for $0 \le i \le d$. Therefore $f_i^*(\theta_i^*)P_{si} = f_0^*(\theta_0^*)P_{s0}$ for $0 \le i \le d$. By this and since $P_{s0} = 1$, we find $P_{si} = f_0^*(\theta_0^*)f_i^*(\theta_i^*)^{-1}$ for $0 \le i \le d$. In other words, for $0 \le i \le d$ the eigenvalue of A_i for F is equal to (1).

(ii)⇒(i): Observe that $\rho^*(A_1^*)$ is symmetrizable by Lemma 5.8, and multiplicity-free since $\{\theta_i^*\}_{i=0}^d$ are mutually distinct. For $0 \le i \le d$ the eigenvalue of A_i for F is P_{si}, and this is equal to $f_0^*(\theta_0^*)f_i^*(\theta_i^*)^{-1}$ by (1). By this and our above comments, for $0 \le i \le d$ the $(s, 0)$-entry of $\rho^*(E_i^*)$ is equal to $|X|^{-1}f_0^*(\theta_0^*)f_i^*(\theta_i^*)^{-1}$. So the $(s, 0)$-entry of $\rho^*(E_i^*)$ times $f_i^*(\theta_i^*)$ is independent of i for $0 \le i \le d$, and this common value is nonzero. Therefore $\rho^*(A_1^*)$ satisfies Theorem 1.3(ii) with $t = 0$. Now by Theorem 1.3, the graph $\Gamma(\rho^*(A_1^*))$ is a bidirected path with endpoints s, 0. Now by Lemma 5.13 $\mathcal{X}$ is Q-polynomial relative to E, and $F = E_s$ is the last primitive idempotent in this Q-polynomial structure. $\square$

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